Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('quinto appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 17 September 2013.

SURNAME.................... NAME.................... MATRICOLA....................

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 1A150 on 20 September 2013 at 14.30.

Duration: 150 minutes

Question 1.

(i) Give the definition of convolution for real-valued functions defined in $\mathbb{R}^N$ and state the Young's Theorem.

(ii) Give the definition of kernel of mollification and the definition of mollifier $A_\delta f$ with step δ of a real-valued function f defined on an open set Ω in $\mathbb{R}^N$.

(iii) Given $\gamma > 0$, we say that a function $f : \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous with exponent $\gamma \in]0, 1]$ if

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty.$$

Prove that if f is Hölder continuous with exponent $\gamma \in]0, 1]$ then also $A_\delta f$ is Hölder continuous with exponent $\gamma \in]0, 1]$.

Answer:

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Question 2.

(i) Give a definition of weak derivative.

(ii) Let $N \geq 2$. Let $f \in C^1(\mathbb{R}^N \setminus \{0\})$ be such that $f \in L^1_{loc}(\mathbb{R}^N)$ and the classical derivatives $\frac{\partial f}{\partial x_i}$ belong to $L^1_{loc}(\mathbb{R}^N)$ for all $i = 1, \dots, N$. Prove that also the weak derivatives $\left(\frac{\partial f}{\partial x_i}\right)_w$ exist for all $i = 1, \dots, N$ and coincide with the corresponding classical derivatives almost everywhere.

(iii) Consider the function $f : \mathbb{R}^N \to \mathbb{R}$ defined almost everywhere by the formula $f(x) = |x| \log |x|$. Discuss existence or non-existence of the weak derivatives $\left(\frac{\partial f}{\partial x_i}\right)_w$ for all $i = 1, \dots N$.

Answer:

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Question 3.

(i) State the Sobolev's Representation formula.

(ii) State the Sobolev Embedding Theorem part 2.

(iii) Discuss briefly an example of an open set for which the Sobolev Embedding Theorem does not hold (it is not required to carry out all computations, but it is required to explain the main idea).

Answer:

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Question 4.

(i) State the Trace Theorem in terms of the appropriate Besov-Nikolskii spaces.

(ii) Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. Give the weak formulation of the classical Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

(iii) Let Ω be a bounded open set in $\mathbb{R}$ of class C^1. Prove in detail that if $v_1, v_2 \in W^{1,2}(\Omega)$ are such that $v_1 - v_2 \in W_0^{1,2}(\Omega)$ then the trace of v_1 is equal the trace of v_2 (almost everywhere in $\partial\Omega$).